On the Steady Motion of an Annular Mass of Rotating Liquid.

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1. The recent investigations of Poincaré* and Professor G. H. Darwin† have drawn attention to the problem of the figures of equilibrium of rotating masses of liquid; and in the present paper it is proposed to consider the steady motion of an annular mass of liquid whose cross-section is approximately circular, and which is rotating as a rigid body under the influence of its own attraction, about an axis through its centre of inertia which is perpendicular to the plane of its central line.

This problem has to some extent been dealt with by Poincaré, who has proved that such figures are possible forms of surfaces of equilibrium; but the subject is capable of further development, and the object of this paper is to show how a solution may be obtained to any degree of approximation by the aid of the Toroidal Function analysis which has been so successfully employed by Mr. W. M. Hicks‡ in his investigations on circular vortex rings.

2. In employing toroidal functions in problems such as the present, I have found it convenient to make use of a modified form of the methods introduced by Mr. Hicks. This method, together with many of the formulae required, will be explained in Chapter XII of my *Treatise on Hydrodynamics*, but for the sake of completeness I shall proceed to give a preliminary sketch.

Writing $\bar{\omega} = (x^2 + y^2)^{\frac{1}{2}}$, and starting with the dipolar transformation

$$z + \iota\bar{\omega} = a \tan \frac{1}{2}(\xi + \iota\eta)$$

instead of the logarithmic form, and putting $C = \cosh\eta$, $c = \cos\xi$, $k = \varepsilon^{-\eta}$,

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* Acta Mathematica, Vol. VII, p. 259.

† Phil. Trans., 1887, p. 379.

‡ Phil. Trans., 1881, p. 609; 1884, p. 161; 1885, p. 725.

it has been shown by Mr. Hicks that the potential of an anchor ring or tore, which is composed of a mass of matter of constant density ρ, may be expressed in the forms

$$V = (C+c)^{\frac{1}{2}} \sum_0^\infty A_n P_n \cos n\xi$$

at an external point, and*

$$V' = -\frac{2}{3}\pi\rho r^2 + (C+c)^{\frac{1}{2}} \sum_0^\infty B_n Q_n \cos n\xi$$

where P_n and Q_n are the two zonal toroidal functions of degree n.

From the formulae given by Mr. Hicks† it appears that P_n and Q_n respectively contain the factors $2k^{-n+\frac{1}{2}}$ and $\pi k^{n+\frac{1}{2}}$, and we shall therefore find it convenient to write $2P_n k^{-n+\frac{1}{2}}$ and $\pi Q_n k^{n+\frac{1}{2}}$ for the functions which he denotes by P_n and Q_n. It will also be shown farther on that if $(C+c)^{\frac{1}{2}}$ be expanded in a series of cosines of multiples of ξ, the coefficient of $\cos n\xi$ will be a rational and integral function of k multiplied by $(2k)^{-\frac{1}{2}}$. We shall therefore write

$$V = (2b)^{\frac{1}{2}}(C+c)^{\frac{1}{2}} \sum A_n P_n (b/k)^{n-\frac{1}{2}} \cos n\xi \tag{1}$$

for the value of the potential at an external point, and

$$V' = -\frac{2}{3}\pi\rho r^2 + (2b)^{\frac{1}{2}}(C+c)^{\frac{1}{2}} \sum B_n Q_n (k/b)^{n+\frac{1}{2}} \cos n\xi \tag{2}$$

for its value at an internal point, where b is the value of k at the surface of the tore.

At the critical circle $\eta = \infty$, and therefore $k = 0$. Now throughout the whole of the present paper the cross-section of the tore will be supposed to be small in comparison with its aperture, and as we shall only require to consider the values of the quantities in the neighbourhood of the tore, the following approximate values of P and Q and their differential coefficients with respect to k, which are denoted by accents, will be sufficient for our purpose, viz:

$$\left.\begin{aligned}
&P_0 = L + \frac{1}{4}k^2(L-1), \quad P_0' = -k^{-1} + \frac{1}{2}k\left(L - \frac{3}{2}\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad P_0'' = k^{-2} + \frac{1}{2}\left(L - \frac{5}{2}\right), \\
&P_1 = 1 + \frac{1}{2}k^2\left(L - \frac{1}{2}\right), \quad P_1' = \frac{1}{2}k\left(L - \frac{3}{2}\right), \quad P_1'' = \frac{1}{2}\left(L - \frac{5}{2}\right), \\
&P_2 = \frac{2}{3}\left(1 + \frac{3}{4}k^2\right), \quad P_2' = k, \qquad\qquad P_2'' = 1,
\end{aligned}\right\} \tag{3}$$

* ρ is supposed to be expressed in astronomical units.

† Phil. Trans., 1884, equations (9) and (10).

where $L = \log 4/k$; also

$$\left.\begin{array}{lll} Q_0 = 1 + \frac{1}{4}k^2, & Q_0' = \frac{1}{2}k, & Q_0'' = \frac{1}{2}, \\ Q_1 = \frac{1}{2}\left(1 + \frac{3}{8}k^2\right), & Q_1' = \frac{3}{8}k, & Q_1'' = \frac{3}{8}, \\ Q_2 = \frac{3}{8}\left(1 + \frac{5}{12}k^2\right), & Q_2' = \frac{5}{16}k, & Q_2'' = \frac{5}{16}, \\ Q_3 = \frac{5}{16} + \qquad , & \text{etc.,} & \text{etc.} \end{array}\right\} \tag{4}$$

3. We shall also require the expansion of r^2, ϖ^2, and $(C + c)^{\frac{1}{2}}$.

Expansion of r^2.

$$\frac{r^2}{a^2} = \frac{C - c}{C + c} = \frac{1 - 2kc + k^2}{1 + 2kc + k^2}$$
$$= 1 - 4kc + 8k^2c^2 + 4k^3(c - 4c^3) - 16k^4(c^2 - 2c^4),$$

higher powers than k^4 being neglected; whence

$$r^2 = a^2\{1 + 4k^2 + 4k^4 - 4(k + 2k^3)\cos\xi + 4(k^2 + 2k^4)\cos 2\xi$$
$$- 4k^3\cos 3\xi + 4k^4\cos 4\xi\}. \tag{5}$$

Expansion of ϖ^2.

$$\frac{\varpi^2}{a^2} = \frac{(1 - k^2)^2}{(1 + 2kc + k^2)^2}$$
$$= 1 - 4kc + 4k^2(3c^2 - 1) + 4k^3(5c - 8c^3) + 8k^4(1 - 9c^2 + 10c^4),$$

whence

$$\varpi^2 = a^2\{1 + 2k^2 + 2k^4 - 4(k + k^3)\cos\xi + 2(3k^2 + 2k^4)\cos 2\xi$$
$$- 8k^3\cos 3\xi + 10k^4\cos 4\xi. \tag{6}$$

Expansion of $(C + c)^{\frac{1}{2}}$.

$$(2k)^{\frac{1}{2}}(C + c)^{\frac{1}{2}} = (1 + 2kc + k^2)^{\frac{1}{2}}$$
$$= 1 + kc + \frac{1}{2}k^2(1 - c^2) - \frac{1}{2}k^3(c - c^3) - \frac{1}{4}k^4\left(\frac{1}{2} - 3c^2 + \frac{5}{2}c^4\right),$$

whence

$$(C + c)^{\frac{1}{2}} = (2k)^{-\frac{1}{2}}\left\{1 + \frac{1}{4}k^2 + \frac{1}{64}k^4 + \left(k - \frac{1}{8}k^3\right)\cos\xi\right.$$
$$\left. - \frac{1}{4}k^2\left(1 - \frac{1}{4}k^2\right)\cos 2\xi + \frac{1}{8}k^3\cos 3\xi - \frac{5}{64}k^4\cos 4\xi\right\}. \tag{7}$$

4. Having obtained these preliminary results, we are in a position to consider the steady motion of the tore.

If Ω be the angular velocity in steady motion, the condition to be satisfied at the surface of the tore is

$$V + \frac{1}{2}\Omega^2(x^2 + y^2) = \text{const.} \tag{8}$$

We are not, however, at liberty to assume that the cross-section is an exact circle in steady motion, and we shall therefore suppose that its equation is

$$\varepsilon^{-\eta} = k = b(1 + \beta_1 \cos\xi + \beta_2 \cos 2\xi + \ldots.) \tag{9}$$

where b is the mean value of k at the surface, which is supposed to be small in comparison with a, the radius of the critical circle; and we shall also assume that β_n is a small quantity of the order b^n. We must therefore first find the potential of an annular mass of matter of uniform density ρ whose cross-section is determined by (9). This is effected by assuming that the potentials at an external and internal point are respectively given by (1) and (2), and determining the coefficients from the consideration that at the surface the values of V and V' must differ by a constant, and that the values of dV/dk and dV'/dk must be equal. We shall thus obtain the values of the coefficients in terms of the β's. The resulting value of V and the surface value of $\bar{\omega}^2$ must then be substituted in (8), and the coefficients of the cosines of multiples of ξ equated to zero. This will determine the values of the β's.

The preceding method will enable us to determine the values of the β's to any degree of approximation that may be desired, but in order to avoid unnecessary complication, the investigation will be confined to the determination of the first term of β_1, which is sufficient to prove the existence of annular figures of equilibrium. From the course of the work it is evident that a higher approximation could be obtained with some additional labour.

We shall find that B_n is of the order b^n, but that A_n is of the order b^{n+2}, and we shall commence with the determination of V' and dV'/dk. In calculating the surface value of the former quantity, it will be unnecessary to proceed farther than the term involving $\cos 2\xi$, or to include terms of a higher order than the second; but in calculating dV'/dk it will be necessary, previously to performing the differentiation, to retain the term $\cos 3\xi$ together with quantities of the third order in the coefficients, since the terms of the third order reduce upon differentiation to terms of the second order.

5. *Calculation of V'.*

From (2), (5) and (7) we obtain

$$V' = -\frac{2}{3}\pi\rho a^2\{1 + 4k^2 - 4(k + 2k^3)\cos\xi + 4k^2\cos 2\xi - 4k^3\cos 3\xi\} + GH', \tag{10}$$

where

$$G = 1 + \frac{1}{4}k^2 + \left(k - \frac{1}{8}k^3\right)\cos\xi - \frac{1}{4}k^2\cos 2\xi + \frac{1}{8}k^3\cos 3\xi, \tag{11}$$

$$H' = B_0Q_0 + B_1Q_1(k/b)\cos\xi + B_2Q_2(k/b)^2\cos 2\xi + B_3Q_3(k/b)^3\cos 3\xi. \tag{12}$$

Omitting terms of a higher order than the second, the surface values of the quantities are

$$-\frac{2}{3}\pi\rho r^2 = -\frac{2}{3}\pi\rho a^2\{1 - 2b\beta_1 + 4b^2 - 4b\cos\xi + (4b^2 - 2b\beta_1)\cos 2\xi\},$$

$$G = 1 + \frac{1}{4}b^2 + \frac{1}{2}b\beta_1 + b\cos\xi + \frac{1}{2}\left(b\beta_1 - \frac{1}{2}b^2\right)\cos 2\xi, \tag{13}$$

$$H' = B_0\left(1 + \frac{1}{4}b^2\right) + \frac{1}{4}B_1\beta_1 + \frac{1}{2}B_1\cos\xi + \left(\frac{1}{4}B_1\beta_1 + \frac{3}{8}B_2\right)\cos 2\xi.$$

Therefore the surface value of V' is

$$V' = \text{const} + \left(\frac{8}{3}\pi\rho a^2 b + B_0 b + \frac{1}{2}B_1\right)\cos\xi$$
$$+ \left\{-\frac{2}{3}\pi\rho a^2(4b^2 - 2b\beta_1) + \frac{1}{4}B_1\beta_1 + \frac{1}{4}B_1 b + \frac{3}{8}B_2\right\}\cos 2\xi. \tag{14}$$

6. *Calculation of dV'/dk.*

From (10), (11) and (12) we obtain

$$\frac{dV'}{dk} = -\frac{8}{3}\pi\rho a^2\{2k - (1 + 6k^2)\cos\xi + 2k\cos 2\xi - 3k^2\cos 3\xi\}$$
$$+ H'\frac{dG}{dk} + G\frac{dH'}{dk}. \tag{15}$$

Now $$\frac{dG}{dk} = \frac{1}{2}k + \left(1 - \frac{3}{8}k^2\right)\cos\xi - \frac{1}{2}k\cos 2\xi + \frac{3}{8}k^2\cos 3\xi. \tag{16}$$

Also

$$H' = B_0\left(1 + \frac{1}{4}k^2\right) + \frac{1}{2}B_1\left(1 + \frac{3}{8}k^2\right)(k/b)\cos\xi + \frac{3}{8}B_2(k/b)^2\cos 2\xi$$
$$+ \frac{5}{16}B_3(k/b)^3\cos 3\xi.$$

Therefore

$$H'\frac{dG}{dk} = \frac{1}{2}B_0k + \frac{1}{4}B_1k/b + \left\{B_0\left(1 - \frac{1}{8}k^2\right) + \frac{1}{8}B_1k^2/b\right.$$
$$\left. + \frac{3}{16}B_2(k/b)^2\right\}\cos\xi + \left(-\frac{1}{2}B_0k + \frac{1}{4}B_1k/b\right)\cos 2\xi, \tag{17}$$

the term involving cos 3ξ being omitted, as it is not required. Also

$$\frac{dH'}{dk} = \frac{1}{2} B_0 k + \frac{1}{2} B_1 b^{-1}\left(1 + \frac{9}{8} k^2\right) \cos \xi + \frac{3}{4} B_2 b^{-2} k \cos 2\xi + \frac{15}{16} B_3 b^{-3} k^2 \cos 3\xi,$$

therefore

$$G\frac{dH'}{dk} = \frac{1}{2} B_0 k + \frac{1}{4} B_1 k/b + \left\{\frac{1}{2} B_1 \left(b^{-1} + \frac{5}{4} k^2/b\right) + \frac{1}{2} B_0 k^2 + \frac{3}{8} B_2 k^2/b^2\right\} \cos \xi + \left(\frac{1}{4} B_1 k/b + \frac{3}{4} B_2 k/b^2\right) \cos 2\xi. \tag{18}$$

Substituting from (17) and (18) in (15), we obtain

$$\begin{aligned}\frac{dV'}{dk} = &-\frac{16}{3}\pi\rho a^2 k + B_0 k + \frac{1}{2} B_1 k/b \\ &+\left\{\frac{8}{3}\pi\rho a^2(1+6k^2) + B_0\left(1+\frac{3}{8}k^2\right) + \frac{1}{2} B_1\left(b^{-1} + \frac{3}{2} k^2/b\right) + \frac{9}{16} B_2 (k/b)^2\right\} \cos\xi \\ &+\left\{-\frac{16}{3}\pi\rho a^2 k - \frac{1}{2} B_0 k + \frac{1}{2} B_1 k/b + \frac{3}{4} B_2 k/b^2\right\} \cos 2\xi.\end{aligned} \tag{19}$$

Putting for k its value from (9), the surface value is

$$\begin{aligned}\frac{dV'}{dk} = &\left(-\frac{16}{3}\pi\rho a^2 b + B_0 b + \frac{1}{2} B_1\right)(1 + \beta_1 \cos\xi) \\ &+\left\{\frac{8}{3}\pi\rho a^2(1+6b^2) + B_0\left(1+\frac{3}{8}b^2\right) + \frac{1}{2} B_1 b^{-1}\left(1 + \frac{3}{2} b^2\right) + \frac{9}{16} B_2\right\} \cos\xi \\ &+\left\{-\frac{16}{3}\pi\rho a^2 b - \frac{1}{2} B_0 b + \frac{1}{2} B_1 + \frac{3}{4} B_2/b\right\}\left(\frac{1}{2}\beta_1 \cos\xi + \cos 2\xi\right).\end{aligned} \tag{20}$$

From (19) or (20) it appears that the most important terms of dV'/dk are of zero order; and when we have calculated the value of dV/dk, it will be found that this circumstance requires that A_n should be of the order b^{n+2}.

7. *Calculation of V.*

Putting

$$H = A_0 P_0 + A_1 (b/k) P_1 \cos\xi + A_2 (b/k)^2 P_2 \cos 2\xi, \tag{21}$$

we have $\qquad V = GH.$

The surface value of G is given by (13); also by (3) at the surface,

$$P_0(k) = P_0(b) + \frac{1}{4}\beta_1^2 - \beta_1 \cos\xi + \left(\frac{1}{4}\beta_1^2 - \beta_2\right)\cos 2\xi,$$

$$(b/k)P_1 \cos\xi = \cos\xi - \frac{1}{2}\beta_1 - \frac{1}{2}\beta_1 \cos 2\xi,$$

$$(b/k)^2 P_2 \cos 2\xi = \frac{2}{3}\cos 2\xi.$$

The surface value of H is therefore

$$H = A_0\left\{L + \frac{1}{4}b^2(L-1) + \frac{1}{4}\beta_1^2\right\} - \frac{1}{2}A_1\beta_1 - (A_0\beta_1 - A_1)\cos\xi$$
$$+ \left\{A_0\left(\frac{1}{4}\beta_1^2 - \beta_2\right) - \frac{1}{2}A_1\beta_1 + \frac{2}{3}A_2\right\}\cos 2\xi. \tag{22}$$

Hence the surface value of V is

$$V = \text{const} + \{A_0(Lb - \beta_1) + A_1\}\cos\xi$$
$$+ \left\{A_0\left(\frac{1}{4}\beta_1^2 - \beta_2 - \frac{1}{2}b\beta_1\right) + \frac{1}{2}A_1(b - \beta_1) + \frac{2}{3}A_2\right\}\cos 2\xi. \tag{23}$$

8. *Calculation of dV/dk.*

We have
$$\frac{dV}{dk} = G\frac{dH}{dk} + H\frac{dG}{dk}.$$

Now from (21) and (3),

$$\frac{dH}{dk} = A_0\left\{-k^{-1} + \frac{1}{2}k\left(L - \frac{3}{2}\right)\right\} - A_1 bk^{-2}\left\{1 + \frac{1}{2}k^2\left(L - \frac{1}{2}\right)\right\}\cos\xi$$
$$+ \frac{1}{2}A_1 b\left(L - \frac{3}{2}\right)\cos\xi + \text{etc.}$$

This has to be multiplied by G whose term of lowest order is unity. Now the constant terms in dV'/dk are of the first order, and therefore the constant terms in dV/dk must also be of the same order, whence A_0/b must be of the first order, and therefore A_0 must be of the second order of small quantities. Similarly it can be shown that all the other A's are of an order two degrees higher than their indices. We can therefore considerably simplify the calculation, since we do not require to retain quantities of a higher order than the second. We thus obtain

$$G\frac{dH}{dk} = -A_0/k - (A_1 b/k^2 + A_0)\cos\xi;$$

also from (16),

$$H\frac{dG}{dk} = A_0 P_0 \cos\xi = A_0 L \cos\xi,$$

whence
$$\frac{dV}{dk} = -A_0/k + (A_0L - A_0 - A_1b/k^2)\cos\xi,$$
and the value of this at the surface is
$$\frac{dV}{dk} = -A_0/b + (A_0\beta_1/b + A_0L - A_0 - A_1/b)\cos\xi; \tag{24}$$
the term involving $\cos 2\xi$ being of the third order, is omitted.

9. We can now obtain the value of V in terms of the β's. From (23) it appears that the coefficients of $\cos\xi$ and $\cos 2\xi$ in V are of the third and fourth orders respectively; and since these quantities are equal to the coefficients of the corresponding terms in V', it follows from (14) that to the lowest order
$$0 = \frac{8}{3}\pi\rho a^2 b + B_0 b + \frac{1}{2}B_1, \tag{25}$$
$$0 = -\frac{2}{3}\pi\rho a^2(4b^2 - 2b\beta_1) + \frac{1}{4}B_1(b+\beta_1) + \frac{3}{8}B_2. \tag{26}$$

Equating the constant terms in dV/dk and dV'/dk, we obtain
$$-\frac{16}{3}\pi\rho a^2 b + B_0 b + \frac{1}{2}B_1 = -A_0/b,$$
whence by (25)
$$A_0 = \frac{8}{3}\pi\rho a^2 b^2. \tag{27}$$

Since the coefficient of $\cos 2\xi$ in dV/dk is of the third order, it follows from (20) that
$$-\frac{16}{3}\pi\rho a^2 b^2 - \frac{1}{2}B_0 b^2 + \frac{1}{2}B_1 b + \frac{3}{4}B_2 = 0. \tag{28}$$

Equating coefficients of $\cos\xi$ in dV/dk and dV'/dk and using (25) and (28), we obtain
$$\begin{aligned}A_0(\beta_1/b + L - 1) - A_1/b &= -8\pi\rho a^2 b\beta_1 + 16\pi\rho a^2 b^2 + \frac{3}{8}B_0 b^2 + \frac{3}{4}B_1 b + \frac{9}{16}B_2\\ &= -8\pi\rho a^2 b\beta_1 + 18\pi\rho a^2 b\end{aligned}$$
by (28), whence
$$A_1 = \frac{2}{3}\pi\rho a^2 b^2(16\beta_1 + 4Lb - 31b). \tag{29}$$

Again from (6),
$$\frac{1}{2}\Omega^2\varpi^2 = \frac{1}{2}\Omega^2 a^2(1 - 4b\cos\xi + \ldots.).$$

Substituting this together with the value of V from (23) in (8), and equating the coefficient of $\cos\xi$ to zero, we obtain
$$\beta_1 = \frac{1}{12}b(31 + 3\Omega^2/\pi\rho b^2 - 8L), \tag{30}$$
which determines β_1.

10. If we take the radius of the critical circle as the unit of length, it might be thought that if we gave to b any small numerical value, the value of Ω would be arbitrary, subject only to the condition that the resulting value of β_1 is a small numerical quantity of the same order as b. This, however, is not the case, for we have tacitly assumed that the pressure does not become negative at any point in the interior of the ring. If the pressure did become negative at any internal point, this would indicate the existence of a hollow space within the ring, and the preceding investigation would be no longer applicable, for the potential at any point in the substance of a ring which contained a hollow would involve the P functions as well as the Q functions; and we must therefore find the condition that the pressure should not become negative within the ring.

Since V' is the attraction potential, the pressure p is determined by the equation

$$p/\rho = V' + \frac{1}{2}\Omega^2(x^2+y^2) + C.$$

Let P be the pressure along the critical circle where $k=0$, then from (10), (11) and (12),

$$P/\rho = -\frac{2}{3}\pi\rho a^2 + B_0 + \frac{1}{2}\Omega^2 a^2 + C.$$

We have already found the conditions that the coefficients of $\cos\xi$, etc., should vanish in p, hence to determine the condition that p should vanish at the free surface, all that we require is the constant term in V'. By §6, this is

$$-\frac{2}{3}\pi\rho a^2(1+4b-2b\beta_1) + B_0 + \frac{1}{2}b\left(B_0 b + \frac{1}{2}B_1\right) + \frac{1}{2}\beta_1\left(B_0 b + \frac{1}{2}B_1\right)$$
$$= -\frac{2}{3}\pi\rho a^2(1+6b^2) + B_0$$

by (25); whence

$$0 = -\frac{2}{3}\pi\rho a^2(1+6b^2) + B_0 + \frac{1}{2}\Omega^2 a^2(1+2b^2-2b\beta_1) + C$$

and $\quad P/\rho = 4\pi\rho a^2 b^2 - \Omega^2 a^2(b^2 - b\beta_1)$.

We must therefore have

$$4\pi\rho - \Omega^2 + \Omega^2\beta_1/b > 0. \tag{31}$$

Let $\lambda = \Omega^2/4\pi\rho$, then substituting the value of β_1 from (30), the condition becomes

$$\lambda^2 - \left(\frac{2}{3}L - \frac{19}{12}\right)b^2\lambda + b^2 > 0. \tag{32}$$

11. As a numerical example, let $b = .1$, then

$$\beta_1 = .0124 + 10\lambda,$$

and if we put $\lambda = .01$, the left-hand side of (32) becomes .9224, and is therefore positive, and therefore

$$\beta_1 = .1124,$$
$$\Omega^2/4\pi\rho = .01$$

are solutions of the problem.

Since (30) may be written in the form

$$\beta_1 = \frac{1}{b}\left(\frac{31}{12}b^2 + \lambda - \frac{2}{3}b^2L\right),$$

it follows that if b is very small, β_1 cannot be of the same order unless λ is very small.

It is also necessary to point out that the preceding analysis proceeds on the assumption that β_2 is of the order b^2, and it is quite possible that if a series of values were assigned to b and λ which satisfy (30) and (32), these values might not satisfy the condition that β_2 should be of the order b^2, or the inequality corresponding to (32) which would be obtained by carrying the approximation one degree higher. This difficulty would not be avoided by calculating the value of β_2, since the same difficulty would exist with regard to β_3. It therefore appears that although toroidal function analysis throws some light on the solution of the problem, it fails to give a perfectly satisfactory approximate solution. The preceding results indicate that for small angular velocities an annular figure exists whose cross-section is approximately circular, but for large angular velocities the cross-section would probably become highly elliptical, and the ring would become flattened; and that if this quantity increased beyond a certain limit, the ring would break up.

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